Primitive “surori”

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Vă propun să calculați următoarele integrale:

1) $\int \sin^2 x dx \quad , x \in \mathbb{R}$

2) $\int \sin^2 x \cdot e^x dx \quad , x \in \mathbb{R}$

3) $\int \frac{\sin x}{\sin x + \cos x} dx \quad , x \in (-\frac{\pi}{4}, \frac{\pi}{4})$

4) $\int \frac{\sin x}{\sqrt{1 + \sin 2x}} dx \quad , x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

5) $\int \frac{e^x + 2x^2 + 5x + 2}{e^x + 4x^2 + 2x + 2} dx \quad , x \in \mathbb{R}$

La rezolvarea acestora nu se pot aplica metodele cunoscute: integrarea prin părți sau metoda schimbării de variabilă. Pentru acestea vom apela la așa zisa integrală “soră”, o integrală care împreună cu cea inițială formează un sistem de tipul:

$$\begin{cases} I + J = \ldots \\ I - J = \ldots \end{cases}$$

Una din aceste două ecuații duce la rezolvarea unei integrale imediate, din tabelul de integrale iar cealaltă presupune rezolvarea unei integrale destul de simplă:

1) $I = \int \sin^2 x dx$, $J = \int \cos^2 x dx$

$$J+I=\int(\cos^2 x+\sin^2 x)dx=\int 1dx=x+c_1$$

$$J-I=\int(\cos^2 x-\sin^2 x)dx=\int \cos 2xdx=\int \frac{(\sin 2x)'}{2}dx=\frac{\sin 2x}{2}+c_2$$

$$\begin{cases} J+I=x+c_1 \\ J-I=\frac{\sin 2x}{2}+c_2 \end{cases}$$

Adunând ecuațiile obținem:

$$2J=x+\frac{\sin 2x}{2}+c_1+c_2 \Rightarrow J=\frac{2x+\sin 2x}{4}+\frac{c_1+c_2}{2}$$

$$I=x+c_1-\frac{2x+\sin 2x}{2}-\frac{c_1+c_2}{2} \Rightarrow I=\frac{2x-\sin 2x}{4}+\frac{c_1-c_2}{2}$$

2) $I=\int \sin^2 x\cdot e^x dx$, $J=\int \cos^2 x\cdot e^x dx$

$$J+I=\int(\sin^2 x+\cos^2 x)e^x dx=\int e^x dx=e^x+c_1$$

$$J-I=\int(\cos^2 x-\sin^2 x)e^x dx=\int \cos 2x\cdot e^x dx$$

Notăm integrala obținută cu A = $\int \cos 2x\cdot e^x dx$ iar pentru rezolvarea ei vom aplica metoda de integrare prin părți:

$$A=\int \cos 2x\cdot (e^x)' dx=\cos 2x\cdot e^x-\int(\cos 2x)'\cdot e^x dx=\cos 2x\cdot e^x-2\int(-\sin 2x)e^x dx=$$
$$=\cos 2x\cdot e^x+2\int \sin 2x e^x dx=\cos 2x\cdot e^x+2\int \sin 2x(e^x)' dx=\cos 2x\cdot e^x+2\sin 2x\cdot e^x-2\int(\sin 2x)' e^x dx$$

$$\begin{cases} J+I=e^x+c_1 \\ J-I=\frac{(\cos 2x+2\sin 2x)e^x}{5}+c_2 \end{cases}$$

$$2J=e^x+\frac{(\cos 2x+2\sin 2x)e^x}{5}+c_2+c_1=\frac{(\cos 2x+2\sin 2x+5)e^x}{5}+c_2+c_1$$

$$J=\frac{(\cos 2x+2\sin 2x+5)e^{x}}{10}+\frac{c_2+c_1}{2}$$

$$I=e^{x}+c_1-\frac{(\cos 2x+2\sin 2x+5)e^{x}}{10}-\frac{c_2+c_1}{2}=\frac{(5-\cos 2x-2\sin 2x)e^{x}}{10}+\frac{c_2-c_1}{2}$$

3) $I=\int\frac{\sin x}{\sin x+\cos x}dx$, $J=\int\frac{\cos x}{\sin x+\cos x}dx$

$$I+J=\int\frac{\sin x+\cos x}{\sin x+\cos x}dx=\int 1dx=x+c_1$$

$$I-J=\int\frac{\sin x-\cos x}{\sin x+\cos x}=\int\frac{\sin x-\sin(\frac{\pi}{2}-x)}{\sin x+\cos x}=\int\frac{2\sin\frac{x-\frac{\pi}{2}+x}{2}\cos\frac{x-\frac{\pi}{2}-x}{2}}{2\sin\frac{x+\frac{\pi}{2}-x}{2}\cos\frac{x-\frac{\pi}{2}+x}{2}}dx=$$

$$=\int\frac{\cos\frac{\pi}{4}\sin(x-\frac{\pi}{4})}{\sin\frac{\pi}{4}\cos(x-\frac{\pi}{4})}dx=\int tg(x-\frac{\pi}{4})dx=-\ln\left|\cos(x-\frac{\pi}{4})\right|+c_2$$

$$\begin{cases} I+J=x+c_1 \\ I-J=-\ln\left|\cos(x-\frac{\pi}{4}\right|+c_2 \end{cases}$$

$$2I=x-\ln\left|\cos(x-\frac{\pi}{4})\right|+c_1+c_2 \Rightarrow I=\frac{x-\ln\left|\cos(x-\frac{\pi}{4})\right|}{2}+\frac{c_1+c_2}{2}$$

$$J=x+c_1-\frac{x-\ln\left|\cos(x-\frac{\pi}{4})\right|}{2}-\frac{c_1+c_2}{2}=\frac{x+\ln\left|\cos(x-\frac{\pi}{4})\right|}{2}+\frac{c_1-c_2}{2}$$

4) $I = \int \frac{\sin x}{\sqrt{1+\sin 2x}}dx$, $J = \int \frac{\cos x}{\sqrt{1+\sin 2x}}dx$

$$I + J = \int \frac{\sin x + \cos x}{\sqrt{1+\sin 2x}}dx = \int \frac{\sqrt{1+\sin 2x}}{\sqrt{1+\sin 2x}}dx = x + c_1$$

$$\sin x + \cos x = \sqrt{1+\sin 2x} \Leftrightarrow (\sin x + \cos x)^2 = \sqrt{1+\sin 2x}^2 \Leftrightarrow$$
$$\Leftrightarrow \sin^2 x + 2\sin x \cdot \cos x + \cos^2 x = 1 + \sin 2x \Rightarrow \sin^2 x + \cos^2 x = 1$$

$$I - J = \int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}}dx = \int \frac{\sqrt{1-\sin 2x}}{\sqrt{1+\sin 2x}}dx = \int \frac{\sqrt{(1-\sin 2x)(1+\sin 2x)}}{(1+\sin 2x)} =$$

$$= \int \frac{\sqrt{1-\sin^2 2x}}{1+\sin 2x}dx = \int \frac{\sqrt{\cos^2 x}}{1+\sin 2x}dx = \int \frac{\left|\cos 2x\right|}{1+\sin 2x}dx$$

$$\left|\cos 2x\right| = +\cos 2x \quad , x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

$$\begin{cases} I+J=x+c_1 \\ I-J=\ln\left|1+\sin 2x\right|+c_2 \end{cases}$$

$$2I = x + \ln\left|1+\sin 2x\right| + c_1 + c_2 \Rightarrow I = \frac{x + \ln\left|1+\sin 2x\right|}{2} + \frac{c_1 + c_2}{2}$$

$$J = x + c_1 - \frac{x + \ln\left|1+\sin 2x\right|}{2} - \frac{c_1 + c_2}{2} = \frac{x - \ln\left|1+\sin 2x\right|}{2} + \frac{c_1 - c_2}{2}$$

5) $\int \frac{e^{x} + 2x^{2} + 5x + 2}{e^{x} + 4x^{2} + 2x + 2}dx$, $J = \int \frac{2x^{2} - 3x}{e^{x} + 4x^{2} + 2x + 2}dx$

$$I + J = \int \frac{e^{x} + 2x^{2} + 5x - 3x + 2}{e^{x} + 4x^{2} + 2x + 2}dx = x + c_{1}$$

$$I - J = \int \frac{e^{x} + 8x + 2}{e^{x} + 4x^{2} + 2x + 2} = \int \frac{(e^{x} + 4x^{2} + 2x + 2)'}{e^{x} + 4x^{2} + 2x + 2}dx = \ln\left|e^{x} + 4x^{2} + 2x + 2\right| + c_{2}$$

$$\begin{cases} I+J=x+c_1 \\ I-J=\ln\left|e^{x}+4x^{2}+2x+2\right|+c_2 \end{cases}$$

$$2I = x + \ln\left|e^{x} + 4x^{2} + 2x + 2\right| + c_{1} + c_{2} \Rightarrow I = \frac{x + \ln\left|e^{x} + 4x^{2} + 2x + 2\right|}{2} + \frac{c_{1} + c_{2}}{2}$$

$$J = x + c_{1} - \frac{x + \ln\left|e^{x} + 4x^{2} + 2x + 2\right|}{2} - \frac{c_{1} + c_{2}}{2} = \frac{x - \ln\left|e^{x} + 4x^{2} + 2x + 2\right|}{2} + \frac{c_{1} - c_{2}}{2}$$